

Generalized power domination in WK-Pyramid Networks

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Abstract

The notion of power domination arises in the context of monitoring an electric power system with as few phase measurement units as possible. The k -power domination number of a graph G is the minimum cardinality of a k -power dominating set (k -PDS) of G . In this paper, we determine the k -power domination number of WK-Pyramid networks, $WKP_{(C,L)}$, for all positive values of k except for $k = C - 1, C \geq 2$, for which we give an upper bound. The k -propagation radius of a graph G is the minimum number of propagation steps needed to monitor the graph G over all minimum k -PDS. We obtain the k -propagation radius of $WKP_{(C,L)}$ in some cases.

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1 Introduction

Power domination is a variation of domination introduced in [2] to address the problem of monitoring electrical networks with *phasor measurement units*. It was described as a graph parameter in [10]. Let $G = (V(G), E(G))$ be a graph that represents an electric power system, where a vertex $v \in V(G)$ represents an electrical node and an edge $e \in E(G)$ represents a transmission line joining two electrical nodes. For a set $S \subseteq V(G)$, the *closed neighbourhood* of S is the union of the closed neighbourhoods of its elements: $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $\langle S \rangle$ denotes the subgraph induced by $S \subseteq V(G)$. A vertex v in a graph is said to dominate its closed neighbourhood $N_G[v]$. A subset $S \subseteq V(G)$ of vertices is a dominating set if $N_G[S] = V(G)$, that is if every vertex in the graph is dominated by some vertex of S . The minimum size of a dominating set in a graph G is called its *domination number*, denoted by $\gamma(G)$.

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In power domination, there is an additional propagation behaviour. Initially, a set S is said to monitor its closed neighbourhood, like in domination. Then, every vertex that is the only unmonitored neighbour of a monitored vertex gets monitored. This possibility of propagation conveys the capacity of deducting the status of a node in an electrical network by applying Kirchhoff laws. It gives to power domination a very different flavour since a vertex may then eventually monitor another vertex far apart as in the case of paths.

The power domination problem was proved to be NP-complete for bipartite graphs and chordal graphs [10]. Linear-time algorithms for this problem were known for trees [10], interval graphs [12] and block graphs [13]. Upper bounds for the power domination number were studied in [15, 14] and closed formulae for the power domination number were also determined for some graphs [8, 7, 9].

Power domination was then generalized in [3] by adding the possibility of propagating up to k vertices, k a non-negative integer. Formally, the set of monitored vertices is then described with the following definition from [3, 5], inspired by what was proposed in [1]:

Definition 1.1 (Monitored vertices). *Let G be a graph, $S \subseteq V(G)$ and $k \geq 0$. The sets $(\mathcal{P}_{G,k}^i(S))_{i \geq 0}$ of vertices monitored by S at step i are defined as follows:*

$$\begin{aligned} \mathcal{P}_{G,k}^0(S) &= N_G[S], \text{ and} \\ \mathcal{P}_{G,k}^{i+1}(S) &= \bigcup \{N_G[v] : v \in \mathcal{P}_{G,k}^i(S) \text{ such that } |N_G[v] \setminus \mathcal{P}_{G,k}^i(S)| \leq k\}. \end{aligned}$$

The second part represents the propagation rule. Since $\mathcal{P}_{G,k}^i(S)$ is always a union of neighbourhoods, $\mathcal{P}_{G,k}^i(S) \subseteq \mathcal{P}_{G,k}^{i+1}(S)$. If $\mathcal{P}_{G,k}^{i_0}(S) = \mathcal{P}_{G,k}^{i_0+1}(S)$ for some i_0 , then $\mathcal{P}_{G,k}^j(S) = \mathcal{P}_{G,k}^{i_0}(S)$ for any $j \geq i_0$. We thus define $\mathcal{P}_{G,k}^\infty(S) = \mathcal{P}_{G,k}^{i_0}(S)$. When the graph G is clear from the context, we will simplify the notations to $\mathcal{P}_k^i(S)$ and $\mathcal{P}_k^\infty(S)$.

Definition 1.2. [3] *A k -power dominating set of G (k -PDS) is a set $S \subseteq V(G)$ such that $\mathcal{P}_{G,k}^\infty(S) = V(G)$. The k -power domination number, $\gamma_{P,k}(G)$, of G is the least cardinality of a k -power dominating set of G . A $\gamma_{P,k}(G)$ -set is a k -PDS in G of cardinality $\gamma_{P,k}(G)$.*

Generalized power domination reduces to the usual power domination when $k = 1$ and to the domination when $k = 0$. In [3], Chang et al. extended several known results for power domination to k -power domination. They gave sharp upper bounds for the generalized power domination number of connected graphs and of connected claw-free $(k + 2)$ -regular graphs. In [6], the authors introduced the k -propagation radius of a graph G , motivated from the studies in [1], as a way to measure the efficiency of a minimum k -PDS. It gives the minimum number of propagation steps needed to monitor the entire graph G over all $\gamma_{P,k}(G)$ -sets. They investigated the relationship between propagation radius and the radius of a graph and also computed the propagation radius of Sierpiński graphs.

Definition 1.3. [6] *The radius of a k -PDS is defined by*

$$\text{rad}_{P,k}(G, S) = 1 + \min \{i : \mathcal{P}_{G,k}^i(S) = V(G)\}.$$

The k -propagation radius of the graph as defined in [6] can be expressed as

$$\text{rad}_{P,k}(G) = \min\{\text{rad}_{P,k}(G, S), S \text{ is a } k\text{-PDS of } G, |S| = \gamma_{P,k}(G)\}.$$

The WK-Pyramid network, an interconnection network based on the WK-recursive mesh [4], was introduced in [11] for massively parallel computers. It has interesting topological characteristics making it suitable for utilization as the base topology of large scale multi-computer systems. It eliminates some drawbacks of the conventional pyramid network, stemming from the fact that the connections within the layers of this network form a WK-recursive mesh. It is of much less network cost than the hypercube, k -ary n -cube and WK-Recursive networks. It also has small average distance and diameter, large connectivity and high degree of scalability and expandability. Because of the desirable properties of this network, it is suitable for medium or large sized networks and also a best alternative for mesh and traditional pyramid interconnection topologies.

For $C, L \in \mathbb{N}$, let $[C] = \{1, \dots, C\}$, $[C]_0 = \{0, \dots, C-1\}$ and $[C]_0^{L-2} = \{a_{L-2}a_{L-3} \dots a_1 : a_i \in [C]_0 \text{ for all } i\}$. An L -level WK-Recursive mesh [11], denoted by $WK_{(C,L)}$, consists of a set of vertices $V(WK_{(C,L)}) = \{(a_L a_{L-1} \dots a_1) : a_i \in [C]_0 \text{ for } i \in [L]\}$. The vertex with address $(a_L a_{L-1} \dots a_1)$ is adjacent

1. to all the vertices with addresses $(a_L a_{L-1} \dots a_2 a_j)$ such that $a_j \in [C]_0$, $a_j \neq a_1$ and
2. to a vertex $(a_L a_{L-1} \dots a_{j+1} a_{j-1} (a_j)^{j-1})$, if there exists one j such that $2 \leq j \leq L$, $a_{j-1} = a_{j-2} = \dots = a_1$ and $a_j \neq a_{j-1}$.

The notation $(a_j)^{j-1}$ denotes that the term a_j is repeated $j-1$ times. Vertices of the form $(\overbrace{a \dots a}^{L \text{ times}})$ are called extreme vertices of $WK_{(C,L)}$. Clearly, $WK_{(C,L)}$ contains C extreme vertices of degree $C-1$ and all the other vertices are of degree C . Note that $WK_{(C,1)} \cong K_C$ ($C \geq 1$), $WK_{(1,L)} \cong K_1$ and $WK_{(2,L)} \cong P_{2^L}$ ($L \geq 1$), where K_C and P_{L+1} denote the complete graph of order C and the path of order 2^L , respectively.

A WK-Pyramid network [11], denoted by $WKP_{(C,L)}$, consists of a set of vertices $V(WKP_{(C,L)}) = \{(r, (a_r a_{r-1} \dots a_1)) : r \in [L], a_i \in [C]_0 \text{ for } i \in [r]\} \cup \{(0, (1))\}$. A vertex with addressing scheme $(r, (a_r a_{r-1} \dots a_1)) \in V(WKP_{(C,L)})$ is called a vertex at level r . The part $(a_r a_{r-1} \dots a_1)$ of the address determines the address of a vertex within the WK-recursive mesh at level r . The vertex $(0, (1))$ is adjacent to every vertex in level 1. A vertex with address $(r, (a_r a_{r-1} \dots a_1))$ at level $r > 0$ is adjacent

1. to vertices $(r, (a_r a_{r-1} \dots a_2 a_j)) \in V(WKP_{(C,L)})$, for $a_j \in [C]_0$, $a_j \neq a_1$
2. to a vertex with address schema $(r, (a_r a_{r-1} \dots a_{j+1} a_{j-1} (a_j)^{j-1}))$, if there exists one j such that $2 \leq j \leq L$, $a_{j-1} = a_{j-2} = \dots = a_1$ and $a_j \neq a_{j-1}$
3. to vertices $(r+1, (a_r a_{r-1} \dots a_2 a_1 a_j))$, for $a_j \in [C]_0$, in level $r+1$ and
4. to a vertex $(r-1, (a_r a_{r-1} \dots a_2))$, in level $r-1$.

All the vertices in level $r > 0$ of $WKP_{(C,L)}$ induce a WK-recursive mesh $WK_{(C,r)}$. Note that $WKP_{(C,1)} \cong K_{C+1}$ ($C \geq 1$), $WKP_{(1,L)} \cong P_{L+1}$ ($L \geq 1$).

Fig. 1 shows the graph $WKP_{(5,2)}$. Vertices of the form $(r, \overbrace{(a \dots a)}^{r \text{ times}})$ are called the extreme vertices of $WKP_{(C,L)}$. The vertex $(0, (1))$ has degree C and at any level except the L^{th} level, the extreme vertices are of degree $2C$ and the other vertices are of degree $2C + 1$. In the L^{th} level, the extreme vertices have degree C and the other vertices have degree $C + 1$.

For $w \in [C]_0^{L-2}$, let $V_w^{C,L} = \{(L, (wij)) \in WKP_{(C,L)} : i, j \in [C]_0\}$ and $G_w^{C,L} = \langle V_w^{C,L} \rangle$, i.e. $G_w^{C,L}$ is the induced subgraph in level L of $WKP_{(C,L)}$. In fact, $G_w^{C,L}$ is isomorphic to $WK_{(C,2)}$ for any $w \in [C]_0^{L-2}$ and any $L \geq 3$ (Fig. 2).

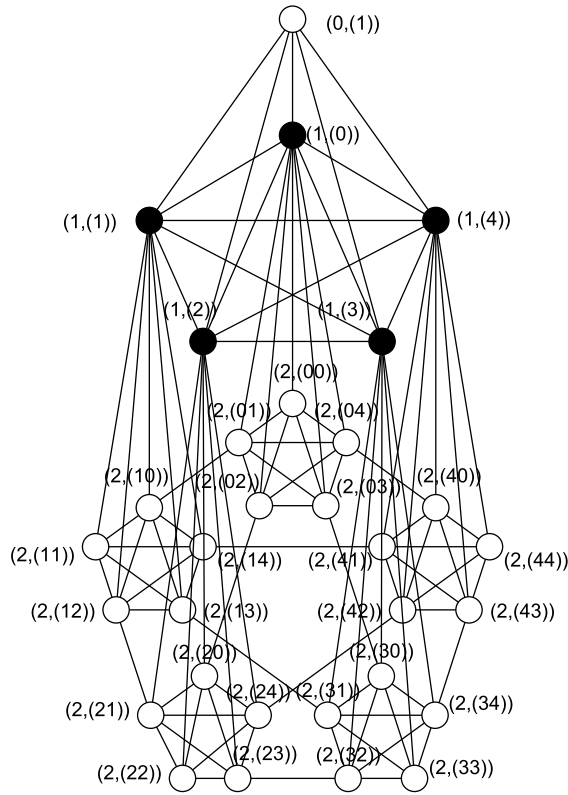


Fig. 1: The graph $WKP_{(5,2)}$

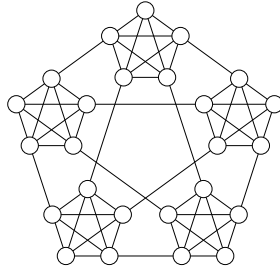


Fig. 2: The induced subgraph $G_w^{5,L}$ of $WKP_{(5,L)}$

The k -power domination number is known only for a few nontrivial families of graphs. In this paper, we determine the k -power domination number of WK-Pyramid network for all positive values of k except for $k = C - 1, C \geq 2$, for which we give an upper bound. This is the first network class with the pyramid structure for which the k -power domination number is studied. We also obtain the k -propagation radius of $WKP_{(C,L)}$ in some cases.

2 k -power domination number of WK-Pyramid network

Theorem 2.1. *Let $C, L, k \geq 1$. If $C = 1$ or $L = 1$ or $k \geq C$, then $\gamma_{P,k}(WKP_{(C,L)}) = 1$.*

Proof. Recall that $WKP_{(C,1)} \cong K_{C+1}$ ($C \geq 1$) and that $WKP_{(1,L)} \cong P_{L+1}$ ($L \geq 1$). Hence $\gamma_{P,k}(G) = 1$ for these graphs G . If $k \geq C$, then take $S = \{(0, (1))\}$. It monitors the vertices in level 1. Since each vertex in level r has exactly C neighbours in its successive level $r + 1$, once the level r is monitored, the vertices in level $r + 1$ get monitored by propagation. This propagation goes on till level L and hence S is a k -PDS of $WKP_{(C,L)}$. \square

We now consider the case when $k \leq C - 1$. We begin with the computation of $\gamma_{P,k}$ for $L = 2$. Let V^1 and V^2 denote the set of vertices of $WKP_{(C,2)}$ in levels 1 and 2 respectively. Let Q_i denote a C -clique induced by the set of vertices $\{(2, (ij)) : j \in [C]_0\}$ for some i . We first obtain the following upper bound.

Lemma 2.2. *For $C \geq 3$ and $k \in [C - 1]$, $\gamma_{P,k}(WKP_{(C,2)}) \leq C - k$.*

Proof. Let $S = \{(1, (i)) : k \leq i \leq C - 1\}$. (The vertices in S are coloured black in Fig. 1.)

$$\text{Then } \mathcal{P}_k^0(S) = \begin{cases} (1, (j)) & ; j \in [C]_0 \\ (2, (ij)) & ; k \leq i \leq C - 1, j \in [C]_0 \\ (0, (1)) \end{cases}$$

$$\mathcal{P}_k^1(S) = \mathcal{P}_k^0(S) \cup \{(2, (ij)) : i \in [k]_0, k \leq j \leq C - 1\} \text{ and}$$

$$\mathcal{P}_k^2(S) = \mathcal{P}^1(S) \cup \{(2, (ij)) : i, j \in [k]_0\} = V(WKP_{(C,2)}).$$

Hence S is a k -PDS, which implies $\gamma_{P,k}(WKP_{(C,2)}) \leq |S| = C - k$. \square

Lemma 2.3. *For $C \geq 3$ and $k \in [C - 2]$, $\gamma_{P,k}(WKP_{(C,2)}) \geq C - k$.*

Proof. Let S be a minimum k -PDS of $WKP_{(C,2)}$. We may assume that $S \subseteq V^1 \cup V^2$.

Claim: $|S \cap (V^1 \cup V^2)| \geq C - k$.

Suppose on the contrary that $|S \cap (V^1 \cup V^2)| \leq C - k - 1$. We consider the case when S contains vertices from both V^1 and V^2 . Assume first that $|S \cap (V^1 \cup V^2)| = C - k - 1$ and that S contains a vertex $(1, (i')) \in V^1$ and the remaining $C - k - 2$ vertices from the C -cliques $Q_{i_1}, \dots, Q_{i_{C-k-2}}$, where $i' \neq i_\ell, \ell \in [C - k - 2]$ such that each of these C -cliques contains exactly one vertex in S . Let Q_ℓ be an arbitrary

clique that does not contain any vertex of S , where $\ell \neq i'$. Let $X = \{(2, (\ell i'))\} \cup \{(2, (\ell i_1)), \dots, (2, (\ell i_{C-k-2}))\}$. Then $\mathcal{P}_k^1(S) \cap Q_\ell = X$. This holds for every $\ell \in L' = [C]_0 \setminus \{i', i_1, \dots, i_{C-k-2}\}$. Thus the set of vertices $J = \{(2, (\ell \ell')) : \ell \in L', \ell' \in L'\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)} \setminus J$ has either 0 or $k+1$ neighbours in J , no vertex from this set J may get monitored later on, which is a contradiction. Assume next that $|S \cap (V^1 \cup V^2)| < C - k - 1$ or that S intersects some C -clique Q_i in more than one vertex. Then we can analogously conclude that not all vertices of Q_ℓ will be monitored. Now, the case when $S \cap V^1 = \emptyset$ or $S \cap V^2 = \emptyset$ can be proved in a similar manner. Hence the claim.

Therefore $\gamma_{P,k}(WKP_{(C,2)}) = |S| = |S \cap (V^1 \cup V^2)| \geq C - k$. \square

Theorem 2.4. For $C \geq 2$ and $k \in [C - 1]$, $\gamma_{P,k}(WKP_{(C,2)}) = C - k$.

Proof. Clearly, $\gamma_{P,1}(WKP_{(2,2)}) = 1$. Let $C \geq 3$. For $k = C - 1$, any vertex in level 1 forms a k -PDS of $WKP_{(C,2)}$. For $k \in [C - 2]$, the result follows from Lemmas 2.2 and 2.3. \square

Lemma 2.5. For $C \geq 3, L \geq 3$ and $k \in [C - 2]$, $\gamma_{P,k}(WKP_{(C,L)}) \leq (C - k - 1)C^{L-2}$.

Proof. In $WKP_{(C,L)}$, the vertices in the L^{th} level induce $WK_{(C,L)}$ which is hamiltonian [11]. Also, by contracting each of the subgraphs $G_w^{C,L}$ into a single vertex, the graph induced by the vertices in level L is isomorphic to $WK_{(C,L-2)}$. Hence, in level L of $WKP_{(C,L)}$, we can arrange the subgraphs of the form $G_w^{C,L}$ into a cycle such that there exists exactly one edge between the consecutive subgraphs. We now construct a set S in such a way that corresponding to each subgraph $G_w^{C,L}$ in level L , S contains one vertex from the neighbour set of $G_w^{C,L}$ in level $L - 1$ (which induces a clique) and $C - k - 2$ additional vertices from $G_w^{C,L}$.

Let $w', w'' \in [C]_0^{L-2}$. Let $G_w^{C,L}, G_{w'}^{C,L}$ and $G_{w''}^{C,L}$ be consecutive subgraphs in the selected hamiltonian order. Let xx' be the edge between $G_w^{C,L}$ and $G_{w'}^{C,L}$, where $x \in G_w^{C,L}, x' \in G_{w'}^{C,L}$ and let $y'y''$ be the edge between $G_{w'}^{C,L}$ and $G_{w''}^{C,L}$, where $y' \in G_{w'}^{C,L}, y'' \in G_{w''}^{C,L}$. Denote $x = (L, (wii))$ and $y' = (L, (w'jj))$ for some i and j , $i \neq j$. Let S contain the vertex $(L - 1, (w'j))$ (which is the neighbour of y' in the $(L - 1)^{th}$ level) and $C - k - 2$ additional vertices from $G_{w'}^{C,L}$ such that no two lying in the same C -clique in $G_{w'}^{C,L}$ and no one lying in the C -clique H in $G_{w'}^{C,L}$ that contains x' . Also $S \cap Q = \emptyset$, where Q is the C -clique in $G_{w'}^{C,L}$ that contains y' . Now, do this in parallel for all the corresponding subgraphs. In particular, the vertex $(L - 1, (wi))$ in the $(L - 1)^{th}$ level corresponding to the vertex x is put into S when considering $G_w^{C,L}$. Thus $C - k$ vertices of H lie in $\mathcal{P}_k^1(S)$: one of these vertices is x' , the other $C - k - 1$ are those vertices of H that have a neighbour in the C -cliques in $G_{w'}^{C,L}$ that contain $C - k - 2$ vertices of S and that have a neighbour in the C -clique Q in $G_{w'}^{C,L}$. Also the neighbour of H in the $(L - 1)^{th}$ level belongs to $\mathcal{P}_k^0(S)$ since $(L - 1, (wi)) \in S$. Hence the remaining k vertices of H lie in $\mathcal{P}_k^2(S)$ and it is straightforward to check that all the vertices of $G_{w'}^{C,L}$ lie in $\mathcal{P}_k^\infty(S)$. In a similar way, every vertex in the L^{th} level is monitored. We know that, for any w , the neighbours of $G_w^{C,L}$ in the $(L - 1)^{th}$ level induce a C -clique. By the construction of S , each C -clique in the $(L - 1)^{th}$ level contains a vertex in S . Thus

we get that all the vertices in levels $L - 1$ and $L - 2$ belong to $\mathcal{P}_k^0(S)$. Now, since each vertex in level $L - 2$ has exactly one neighbour in its preceeding level, vertices in the $(L - 3)^{rd}$ level are monitored by propagation. This propagation continues to the preceeding levels and hence the whole graph gets monitored. Thus we conclude that S is a k -PDS. Since each subgraph $G_w^{C,L}$ contains $C - k - 1$ vertices of S , $|S| \leq (C - k - 1)C^{L-2}$. \square

Lemma 2.6. For $C \geq 3, L \geq 3$ and $k \in [C - 2]$, $\gamma_{P,k}(WKP_{(C,L)}) \geq (C - k - 1)C^{L-2}$.

Proof. Let S be a minimum k -PDS of $WKP_{(C,L)}$ and $w \in [C]_0^{L-2}$. Denote $V_w^{C,L-1} = \{(L - 1, (wi)) \in WKP_{(C,L)} : i \in [C]_0\}$.

Claim: $|S \cap (V_w^{C,L} \cup V_w^{C,L-1})| \geq C - k - 1$.

Suppose on the contrary that $|S \cap (V_w^{C,L} \cup V_w^{C,L-1})| \leq C - k - 2$. Consider the case when $S \cap V_w^{C,L-1} = \emptyset$. Then $|S \cap V_w^{C,L}| \leq C - k - 2$. Assume first that $|S \cap V_w^{C,L}| = C - k - 2$. Let H_i be a C -clique in $G_w^{C,L}$, i.e. H_i is induced by the set of vertices $\{(L, (wij)) \in WKP_{(C,L)} : j \in [C]_0\}$ for some i . Assume that S has exactly one vertex in C -cliques H_i for $i \in \{i_1, \dots, i_{C-k-2}\}$. Then $S \cap V(H_{i'}) = \emptyset$ holds for other $k + 2$ coordinates i' . Let H_ℓ be an arbitrary such clique in $G_w^{C,L}$ that does not contain any vertex of S . Let $X = \{(L, (wli_1)), \dots, (L, (wli_{C-k-2}))\} \cup \{(L, (w\ell))\}$. Then $\mathcal{P}_k^1(S) \cap H_\ell \subseteq X$. This holds for every $\ell \in L' = [C]_0 \setminus \{i_1, \dots, i_{C-k-2}\}$. Thus the set of vertices $\{(L, (w\ell')) : \ell \in L', \ell' \in L', \ell \neq \ell'\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)}$ has either 0 or $k + 1$ neighbours in this set, no vertex from this set may get monitored later on, a contradiction. Assume next that $|S \cap V_w^{C,L}| < C - k - 2$ or that S intersects some C -clique H_i in more than one vertex. Then we can analogously conclude that not all vertices of H_ℓ will be monitored. Thus the case when $S \cap V_w^{C,L-1} = \emptyset$ is not possible.

Now suppose that $S \cap V_w^{C,L-1} \neq \emptyset$. Assume first that $|S \cap (V_w^{C,L} \cup V_w^{C,L-1})| = C - k - 2$ and that S contains a vertex $(L - 1, (wi')) \in V_w^{C,L-1}$ and the remaining $C - k - 3$ vertices from the C -cliques $H_{i_1}, \dots, H_{i_{C-k-3}}$, where $i' \neq i_\ell, \ell \in [C - k - 3]$ such that each of these C -cliques contains exactly one vertex in S . Let H_ℓ be an arbitrary clique in $G_w^{C,L}$ that does not contain any vertex of S , where $\ell \neq i'$. Let $X = \{(L, (wli'))\} \cup \{(L, (w\ell))\} \cup \{(L, (wli_1)), \dots, (L, (wli_{C-k-3}))\}$. Then $\mathcal{P}_k^1(S) \cap H_\ell \subseteq X$. This holds for every $\ell \in L' = [C]_0 \setminus \{i', i_1, \dots, i_{C-k-3}\}$. Thus the set of vertices $\{(L, (w\ell')) : \ell \in L', \ell' \in L', \ell \neq \ell'\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)}$ has either 0 or $k + 1$ neighbours in this set, no vertex from this set may get monitored later on, which is a contradiction. Assume next that $|S \cap (V_w^{C,L} \cup V_w^{C,L-1})| < C - k - 2$ or that S intersects some C -clique H_i in more than one vertex. Then we can analogously conclude that not all vertices of H_ℓ will be monitored. Hence the claim. Therefore, $|S \cap (V_w^{C,L} \cup V_w^{C,L-1})| \geq C - k - 1$, i.e. $|S \cap (V(G_w^{C,L}) \cup N_{L-1}(G_w^{C,L}))| \geq C - k - 1$, where $N_{L-1}(G_w^{C,L})$ is the set of neighbours of $G_w^{C,L}$ in the $(L - 1)^{th}$ level. Hence corresponding to each $G_w^{C,L}$ in the L^{th} level, we get at least $C - k - 1$ vertices in S .

Hence $|S| \geq \sum_{w \in [C]_0^{L-2}} (C - k - 1) = (C - k - 1)C^{L-2}$. \square

Theorem 2.7. For $C \geq 3, L \geq 3$ and $k \in [C - 2]$, $\gamma_{P,k}(WKP_{(C,L)}) = (C - k - 1)C^{L-2}$.

Proof. Follows from Lemmas 2.5 and 2.6. □

Thus we have the following consolidated result:

Let $C, L, k \geq 1$. Then

$$\gamma_{P,k}(WKP_{(C,L)}) = \begin{cases} 1; & C = 1 \text{ or } L = 1 \text{ or } k \geq C, \\ C - k; & L = 2, C \geq 2, k \in [C - 1], \\ (C - k - 1)C^{L-2}; & L \geq 3, C \geq 3, k \in [C - 2]. \end{cases}$$

For $k = C - 1$, $C \geq 2$ and $L \geq 3$ we prove the following upper bound.

Theorem 2.8. For $C \geq 2$, $L \geq 3$, $\gamma_{P,C-1}(WKP_{(C,L)}) \leq \lceil \frac{L+1}{3} \rceil$.

Proof. We consider three cases.

Case 1. $L = 3m$

$$S = \{ \bigcup_{i=1}^m (3i - 1, (0)^{3i-1}) \} \cup \{ (0, (1)) \}.$$

$$\text{Here, } |S| = m + 1. \text{ Also, } \lceil \frac{L+1}{3} \rceil = \lceil \frac{(3m)+1}{3} \rceil = m + 1.$$

Case 2. $L = 3m + 1$

$$S = \{ \bigcup_{i=1}^m (3i, (0)^{3i}) \} \cup \{ (1, (0)) \}.$$

$$\text{Here, } |S| = m + 1. \text{ Also, } \lceil \frac{L+1}{3} \rceil = \lceil \frac{(3m+1)+1}{3} \rceil = m + 1.$$

Case 3. $L = 3m + 2$

$$S = \{ \bigcup_{i=1}^{m+1} (3i - 2, (0)^{3i-2}) \}.$$

$$\text{Here, } |S| = m + 1. \text{ Also, } \lceil \frac{L+1}{3} \rceil = \lceil \frac{(3m+2)+1}{3} \rceil = m + 1.$$

In each case, $\mathcal{P}_{C-1}^\infty(S) = V(WKP_{(C,L)})$ and thus S is a k -PDS of order $\lceil \frac{L+1}{3} \rceil$. Hence $\gamma_{P,C-1}(WKP_{(C,L)}) \leq \lceil \frac{L+1}{3} \rceil$. □

3 Propagation radius of WK-Pyramid network

In this section, we determine the k -propagation radius of $WKP_{(C,L)}$ for $C \geq 1$ and $L = 1, 2$. If $L = 1$, the graph is a complete graph and the propagation radius is 1. If $C = 1$, $\text{rad}_{P,k}(WKP_{(1,L)}) = \text{rad}_{P,k}(P_{L+1}) = \lfloor \frac{L+1}{2} \rfloor$.

Lemma 3.1. Let $C \geq 3$ and $k \in [C - 1]$ and S be a minimum k -PDS of $WKP_{(C,2)}$. Then $S \cap V^1 \neq \phi$.

Proof. Suppose that $S \cap V^1 = \phi$. Consider the case when $(0, (1)) \notin S$. Then by Theorem 2.4, $|S \cap V^2| = C - k$. Assume first that S has exactly one vertex in C -cliques Q_i for $i \in \{i_1, \dots, i_{C-k}\}$. Then $S \cap V(Q_{i'}) = \phi$ for k coordinates i' . Let Q_ℓ be an arbitrary such subgraph. Let $X = \{(2, (\ell i_1)), \dots, (2, (\ell i_{C-k}))\}$. Then $\mathcal{P}_k^1(S) \cap V(Q_\ell) = X$ and $\mathcal{P}_k^1(S) \cap \langle V^1 \rangle = \{(1, i_1), \dots, (1, i_{C-k})\}$. This holds for any $\ell \in L = [C]_0 \setminus \{i_1, \dots, i_{C-k}\}$. Therefore the set of vertices $K = \{(2, (ij)) : i, j \in L\} \cup \{(1, (i)) : i \in L\} \cup \{(0, (1))\}$ has an empty intersection with

$\mathcal{P}_k^1(S)$. Since every vertex of $WKP_{(C,2)} \setminus K$ has either 0 or $k+1$ neighbours in K , no vertex from this set may get monitored later on, a contradiction. The case when $(0, (1)) \in S$ or that S intersects some Q_i in more than one vertex can be proved analogously. \square

Theorem 3.2. *Let $C \geq 2$ and $k \geq 1$. Then*

$$\text{rad}_{P,k}(WKP_{(C,2)}) = \begin{cases} 2; & k \geq C, \\ 3; & k \in [C-1]. \end{cases}$$

Proof. For $k \geq C$, $\gamma_{P,k}(WKP_{(C,2)}) = 1$, by Theorem 2.1 and observe that $\gamma(WKP_{(C,2)}) > 1$. Therefore, $\text{rad}_{P,k}(WKP_{(C,2)}) \geq 2$ (see [6, Proposition 4.1]). And, for the set $S = \{(0, (1))\}$, we get that $\mathcal{P}_k^0(S) = S \cup V^1$ and $\mathcal{P}_k^1(S) = V(WKP_{(C,2)})$. Now let $k \in [C-1]$. For $C = 2$, the result easily follows. Let $C \geq 3$. By Theorem 2.4, $\gamma_{P,k}(WKP_{(C,2)}) = C - k$ and therefore by Lemma 3.1, $|S \cap V^2| \leq C - k - 1$ for every minimum k -PDS S . Then there exists at least $k+1$ C -cliques Q_i not containing any vertex of S . Let $Q_{i'}$ be an arbitrary clique such that $S \cap V(Q_{i'}) = \emptyset$ and $(1, (i')) \notin S$. We prove that the vertex $(2, (i'i'))$ is not in $\mathcal{P}_k^1(S)$. Clearly, $(2, (i'i')) \notin \mathcal{P}_k^0(S)$. Moreover, $|V(Q_{i'}) \cap \mathcal{P}_k^0(S)| \leq C - k - 1$ and $|V(Q_{i'}) \setminus \mathcal{P}_k^0(S)| \geq k+1$. Therefore any neighbour of $(2, (i'i'))$ in $Q_{i'}$ is adjacent to more than k unmonitored vertices preventing any propagation to this vertex at this step. Also, since $(1, (i'))$ has more than k unmonitored vertices as its neighbours, $(2, (i'i'))$ cannot be monitored by $(1, (i'))$ at this stage. Hence $\text{rad}_{P,k}(WKP_{(C,2)}) \geq 3$. Also, by Lemma 2.2, $\text{rad}_{P,k}(WKP_{(C,2)}) \leq 3$. \square

Note: For $C, L \geq 3$ and $k \geq 1$, by observing the propagation behaviour described in the proof of Theorem 2.1 and Lemma 2.5, one can obtain that $\text{rad}_{P,k}(WKP_{(C,L)}) \leq L$ if $k \geq C$ and $\text{rad}_{P,k}(WKP_{(C,L)}) \leq \max\{5, L-1\}$ if $k \in [C-2]$.

4 Conclusion

In this paper, we have determined the k -power domination number of WK-Pyramid networks, $WKP_{(C,L)}$, for all positive values of k except for $k = C-1, C \geq 2$, for which we give an upper bound. We also obtain the k -propagation radius of $WKP_{(C,L)}$ in some cases. The k -power domination number of other pyramid networks such as grid pyramids, torus pyramids can be studied in future.

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